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SOLUTION OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS BY THE METHOD OF EXPANSION IN POWER SERIES ON QUICK-RESPONSE COMPUTERS

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SUMMARY

This paper considers the question of utilization of the method of expansion in series by powers for the solution of systems of ordinary differential equations. The fundamental principles are expounded for the programming of actions with series. The possibilities are discussed of practical application of a specialized program for actions with series during the solution of certain problems by means of a computer.

*
* *

1. Solution of a System of Ordinary Differentiel Equations by the Method of Expansion in Power Series

When resolving systems of ordinary differential equations one often utilizes also the method of expansion of the functions sought for in power series of which the coefficients are determined from systems of algebraic equations. The series obtained constitute the solution of a system of differential equations within the smallest series convergence interval. The difficulties with which the determination of coefficients is beset at great powers of the independent variable are a shortcoming of the given method; this, however, is apparently the only possibility of finding a numerical solution. Thus, expanding the function searched for in series by powers, one may seek the solution in the neighborhood of singular points of some form, while this can not be done at numerical integration by steps.

Let a system be given of ordinary nonlinear differential equations, consisting of $\underline{k}$ equations of first order:

$$\begin{aligned} &F_1[x, y_1, \ldots, y_k, y_1', \ldots, y_k'] = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &F_k[x, y_1, \ldots, y_k, y_1', \ldots, y_k'] = 0. \end{aligned} \tag{1}$$

* per request.

It should be noted that the principles of programming described in this paper are also applicable to equations of higher order; however, the author hesitates in formulating the condition of applicability of these principles to be imposed on the system of equations.

Let us represent the functions sought for in the neighborhood of the point x_0 in the form of series with yet indeterminate coefficients (method of indeterminate coefficients):

$$\begin{aligned} y_1[x] &= y_{10} + y_{11}(x - x_0) + \ldots + y_{1n}(x - x_0)^n + \ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_k[x] &= y_{k0} + y_{k1}(x - x_0) + \ldots + y_{kn}(x - x_0)^n + \ldots . \end{aligned} \tag{2}$$

We shall substitute (2) into (1) and group the terms with identical powers $(x - x_0)$:

$$\begin{aligned} F_{10} + F_{11}(x - x_0) + \ldots + F_{1n}(x - x_0)^n + \ldots &\equiv 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_{k0} + F_{k1}(x - x_0) + \ldots + F_{kn}(x - x_0)^n + \ldots &\equiv 0. \end{aligned}$$

The obtained system of identities is idempotent to the system of algebraic equations

$$\begin{aligned} F_{10} = F_{11} = \ldots = F_{1n} = \ldots &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ F_{k0} = F_{k1} = \ldots = F_{kn} = \ldots &= 0. \end{aligned}$$

In the following we shall consider only those systems (1)in which upon the indicated transformations F_{in} depend only on y_{js}, where $s \leqslant n, 1 \leqslant i, j \leqslant k$ (a fairly broad class of systems of equations satisfy the last condition). In this case we may break down the infinite system of algebraic equations into a set of systems by k equations in each one, and subsequently find coefficients of still greater ordinal number:

$$F_{10}[y_{10}, \ldots, y_{k0}] = \ldots = F_{k0}[y_{10}, \ldots, y_{k0}] = 0, \tag{3^0}$$

$$F_{11}[y_{10}, \ldots, y_{k0}, y_{11}, \ldots, y_{k1}] = \ldots = F_{k1}[y_{10}, \ldots, y_{k0}, y_{11}, \ldots, y_{k1}) = 0. \tag{3^1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\begin{aligned} F_{1n}[y_{10}, \ldots, y_{k0}, y_{11}, \ldots, y_{k1}, \ldots, y_{1n}, \ldots y_{kn}] = \ldots = F_{kn}[y_{10}, \ldots \\ \ldots, y_{k0}, y_{11}, \ldots, y_{k1}, \ldots, y_{1n}, \ldots, y_{kn}] = 0, \end{aligned} \tag{3^n}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In the interval of series convergence the solution of the system (3) is idempotent to the solution of system (1).

If the point x_0 is not singular, system (3^0) is a system of identities and y_{i0}, where $1 \leqslant i \leqslant k$, must be assigned as initial conditions. If the point x_0 is singular (we have in mind a singular point in the neighborhood of which the solution may be expanded in series along certain exclusive directions), at least part of equations of system (3^0) is not fulfilled identically.

In this case some of the coefficients y_{i0} are given as initial conditions, the remaining ones being determined from (3^0). Beginning with (3^1), the coefficients of the series are found from the corresponding systems of equations, whereupon it is indifferent whether or not the point x_0 is singular.

As a rule, Eqs.(3^i), where $i > 1$, are linear relative to y_{ij}, where $1 \leqslant j \leqslant k$. In order to facilitate the expounding we shall precisely deal with such systems. The last condition is not essential. If it is not fulfilled, all the following is valid beginning with a certain system (3^i), where $i > 1$.

The transition from the system (1) of differential equations to the system)3) of algebraic equations is extremely cumbersome. At great i the analytical expressions for the coefficients y_{ij} are very complex, and only for certain particular cases one may succeed in finding recurrent dependences facilitating the computation of the coefficients. The contemporary computing technology offers us new possibilities for the solution of systems of differential equations. Quick-response computers may not only conduct calculations according to before-hand programmed formulas, but also assume the task of composing the formulas and the computation circuits.

A series of national as well as foreign works have been devoted to the conducting of analytical calculations with the aid of electronic computers. We shall note, in particular, the work [1], devoted to the working out of layout symbolics.

2. Representation of Series in the Storage of the Computer

We shall bring forth the list of operations required for the solution of the system (1) of differential equations by the method of expansion in power series: 1) representation of the solution sought for in the form of series with unknown coefficients; 2) substitution of the series into the system of differential equations and performance of all actions foreseen by the left-hand parts; 3) grouping of terms with identical powers $(x - x_0)$ and obtaining the system of algebraic equations; 4) solution of the system of algebraic equations, determination of the coefficients.

The coefficients y_{ij}, where $1 \leqslant i \leqslant k$, are found in sequence: first y_{i1}, then y_{i2} and so forth (we consider that y_{i0} are determined). Assume that all the y_{ij} are known for $0 \leqslant j \leqslant (m-1)$. We are required to determine y_{im}. To that effect there is no requirement of retaining during actions with the series of terms with powers $(x - x_0)$ greater than m. Each action with series should be attended by grouping the terms with identical powers $(x - x_0)$, i. e. 2) and 3) must be fulfilled simultaneously. As a result of the operations performed all Eqs.(3^j), where $0 \leqslant j \leqslant (m-1)$, will be satisfied identically, while (3^m) will be representing the system of alebraic linear equations for the determination of y_{im}.

We shall write the functions $y_i[x]$ sought for in the form of polynomials:

$$
\begin{aligned}
y_1[x] &= y_{10} + y_{11}(x - x_0) + \ldots + y_{1(m-1)}(x - x_0)^{m-1} + y_{1m}(x - x_0)^m,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
y_k[x] &= y_{k0} + y_{k1}(x - x_0) + \ldots + y_{k(m-1)}(x - x_0)^{m-1} + y_{km}(x - x_0)^m;
\end{aligned} \tag{4}
$$

here y_{ij} are known for $j \leq (m - 1)$, and y_{im} are unknown. During actions with polynomials (upon grouping and rejecting the powers $(x - x_0) > m$) we obtain each time certain intermediate polynomials of m-th power:

$$
\begin{aligned}
A_m[x] = A_0 + A_1(x - x_0) + \ldots + A_{m-1}(x - x_0)^{m-1} + \\
+ (A_{m0} + A_{m1}y_{1m} + \ldots + A_{mk}y_{km})(x - x_0)^m,
\end{aligned} \tag{5}
$$

where $A_0, A_1, \ldots, A_{m-1}, A_{m0}, A_{m1}, \ldots, A_{mk}$ are known and $y_{1m}, \ldots, y_{km}$ are unknown. We shall represent the polynomials $A_m[x]$ in the storage of the computer in the form of a sequence of codes:

$$
\begin{gathered}
[a] = A_{m1}, \ldots, [a + k - 1] = A_{mk},\ [a + k] = A_0,\\
[a + k + 1] = A_1,\ \ldots,\ [a + k + m - 1] = A_{m-1},\ [a + k + m] = A_{m0},\\
[a + k + m + 1] = 0, \ldots, [a + k + M] = 0.
\end{gathered} \tag{6}
$$

Here M is the maximum power $(x - x_0)$, to which we wish to limit ourselves. For the functions $y_i(x)$ searched for the sequence of codes has the form

$$
\begin{gathered}
[b_i] = 0, \ldots, [b_i + i - 1] = 1, \ldots, [b_i + k - 1] = 0,\\
[b_i + k] = y_{i0},\ [b_i + k + 1] = y_{i1}, \ldots, [b_i + k + m - 1] = y_{i(m-1)},\\
[b_i + k + m] = 0,\ [b_i + k + m + 1] = 0, \ldots, [b_i + k + M] = 0.
\end{gathered} \tag{7}
$$

Upon completion of all the operations 2) and 3), we shall obtain the sequence of codes for the left-hand part of the system of Eqs.(1):

$$
\begin{gathered}
[c_i] = F_{im1}, \ \ldots, \ [c_i + k - 1] = F_{imk},\ [c_i + k] = F_{i0} \equiv 0,\\
[c_i + k + 1] = F_{i1} \equiv 0, \ \ldots, \ [c_i + k + m - 1] = F_{i(m-1)} \equiv 0,\\
[c_i + k + m] = F_{im0},\ [c_i + k + m + 1] = 0, \ \ldots, \ [c_i + k + M] = 0.
\end{gathered} \tag{8}
$$

Zeros form in the locations $(c_i + k)$ to $(c_i + k + m - 1)$, while in the locations from (c_i) to $(c_i + k - 1)$ and $(c_i + k + m)$ there are formed the coefficients of linear algebraic equations:

$$
\begin{aligned}
F_{1m0} + F_{1m1}y_{1m} + \ldots + F_{1mk}y_{km} = 0,\\
\ldots\ldots\ldots\ldots\ldots\\
F_{km0} + F_{km1}y_{1m} + \ldots + F_{kmk}y_{km} = 0.
\end{aligned} \tag{9}
$$

Upon solution of system (9) y_{im} are dispatched into locations $(b_i + k + m)$. Subsequently, all the operations are repeated for the polynomials of $(m + 1)$-th power: we find $y_{i(m+1)}$ and so forth.

3. Specialized Program for Action with Series

The shaping (8) consists of sequential operations over polynomials of m-th power; the result of each operation is also a polynomial of m-th power of $(x - x_0)$, which appears in the computer storage in the form of sequence of codes (6). We propose below an assortment of standard operations, of which the combination allows us the programming of computation (8) for a wide class of differential equations:

$$
\begin{aligned}
&1)\ A_m[x] + B_m[x] &&= C_m[x],\\
&2)\ A_m[x] - B_m[x] &&= C_m[x],\\
&3)\ A_m[x]r &&= C_m[x],\\
&4)\ A_m[x](x - x_0)^p &&= C_m[x],\\
&5)\ A_m[x]B_m[x] &&= C_m[x],\\
&6)\ (x - x_0)\frac{d}{dx}A_m[x] &&= C_m[x],\\
&7)\ \sin A_m[x] &&= C_m[x],\\
&8)\ \cos A_m[x] &&= C_m[x],\\
&9)\ \exp\{A_m[x]\} &&= C_m[x],\\
&10)\ \ln A_m[x] &&= C_m[x],\\
&11)\ \arcsin A_m[x] &&= C_m[x],\\
&12)\ \mathrm{tg}\, A_m[x] &&= C_m[x],\\
&13)\ \mathrm{arc\,tg}\, A_m[x] &&= C_m[x];
\end{aligned}
$$

here $\underline{r}$ is a constant number, $\underline{p}$ is a whole number, $p \geq 1$.

A special block is programmed for each of these 13 operations, which fulfills the given standard operation. When manupulating the block, the required information is fed to it: the number of equations $\underline{k}$, the power $\underline{m}$ of polynomials, the start addresses of code sequences for polynomials $\langle A_{m1}\rangle$, $\langle B_{m1}\rangle$, $\langle C_{m1}\rangle$, the address $\underline{r}$ for the block 3), the number $\underline{p}$ for the block 4).

Let us pause briefly on principles of block construction. The blocks 1) - 6) are formed elementarily and call for no comments. Note only that in systems which transform to the form (3), the differentiation is always attended by multiplication by $(x - x_0)$ and hence the specification of operation 6).

In the remaining blocks we are required to find the function of the series, that is, a complex functions from the independent variable $\phi = \Phi\{v[x]\}$;

$$\varphi[x] = \varphi\Big|_{x=x_0} + \frac{d\varphi}{dx}\Big|_{x=x_0}(x - x_0) + \ldots + \frac{1}{n!}\frac{d^n\varphi}{dx^n}\Big|_{x=x_0}(x - x_0)^n + \ \ldots$$

For great $\underline{n}$ the form of derivatives $d^n\phi/dx^n$ is quite complex:

$$
\begin{aligned}
\frac{d\varphi}{dx} &= \frac{d\Phi}{dv}\frac{dv}{dx},\\
\frac{d^2\varphi}{dx^2} &= \frac{d^2\Phi}{dv^2}\left(\frac{dv}{dx}\right)^2 + \frac{d\Phi}{dv}\frac{d^2v}{dx^2},\\
\frac{d^3\varphi}{dx^3} &= \frac{d^3\Phi}{dv^3}\left(\frac{dv}{dx}\right)^3 + 3\frac{d^2\Phi}{dv^2}\frac{dv}{dx}\frac{d^2v}{dx^2} + \frac{d\Phi}{dv}\frac{d^3v}{dx^3},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{d^n\varphi}{dx^n} &= \frac{d^n\Phi}{dv^n}\left(\frac{dv}{dx}\right)^n + \ldots + \frac{d^l\Phi}{dv^l}V\left[\frac{dv}{dx}, \frac{d^2v}{dx^2}, \ldots, \frac{d^{n-l+1}v}{dx^{n-l+1}}\right] + \ldots\\
&\quad \ldots + \frac{d\Phi}{dv}\frac{d^nv}{dx^n}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

The functions V have a cumbersome character, and because of their form the recurrent relations, practical for programmings, could not be obtained.

When programming blocks 7) - 11), another approach was utilized. First we shall expand ϕ in series by v in the neighborhood of v_0, which is the value of v for $x = x_0$:

$$\varphi[v] = \Phi|_{v=v_0} + \frac{d\Phi}{dv}\bigg|_{v=v_0} (v - v_0) + \ldots + \frac{1}{n!}\frac{d^n\Phi}{dv^n}\bigg|_{v=v_0} (v - v_0)^n + \ldots. \quad (10)$$

The values of $d^n \Phi/dv^n$ for the functions of sin v, cos v, e^v, ln v, arc sin v were computed by simple formulas or by recurring links, practical for the programming; $(v - v_0)$ is a well known series of v[x] without free term:

$$v - v_0 = v_1(x - x_0) + \ldots + v_n(x - x_0)^n + \ldots.$$

When forming $(v - v_0)^n$, the block 5).

The blocks 7) - 11) consist of two parts. In the first part, different for the various blocks, we compute the derivatives d^n/dv^n. In the second part, common for the blocks 7) - 11), actions are performed by formula (10), the terms with identical powers $(x - x_0)$ are grouped and the powers $(x - x_0)$ greater than m are rejected.

The blocks 12), 13) are formed as combinations of blocks described earlier and the computations are conducted by the well known trigonometric formulas

$$\operatorname{tg} v = \frac{\sin v}{\cos v}, \qquad \operatorname{arc\,tg} v = \arcsin \frac{v}{\sqrt{1 + v^2}}.$$

Besides the standard blocks 1) - 13), the decipherable and tuning sub-programs must be include in the specialized program. The decipherable sub-program allows us to organize the conversion to standard blocks with the aid of one code (for three-code computers), containing a part of the required information: the initial addresses of sequences of polynomial codes and the number of the standard block (and also for the block 3) and p for the block 4)). The remainder of the required information, common for all blocks, that is, the number of equations, k, the power of polynomials, m (or the ordinal number of the coefficients being computed) is communicated to standard blocks by the tuning sub-program. It is necessary to foresee in the basic (operational) program the solution of the system of linear algebraic equations (9) and the substitution of the coefficients found into (7).

The operational program for the solution of the ordinary differential equations by the method of expansion in power series (in the presence in computer storage of the specialized program for action with series) is in its size not larger than the standard operational program for the numerical in te ration of the very same system of differential equations by steps. When composing the operational program, one must write only the transformation codes to standard blocks, fores e the transformations to deciphering, tuning sub-programs and to the program for the solution of the system of algebraic equations, organizing the cycle by m. As a result, the computer determines the numerical values of the coefficients for the series sought for of any ordinal number m (the programmer not writing formulas for y_{im}).

4. Utilization of the Specialized Program for the Solution of Certain Problems.

The author composed a specialized program for the electronic computer. It was utilized during the solution of a system of two differential equations of second order of a rather cumbersome form (see [2]). At $x = x_0 = 0$, the system of equations had a singularity, namely, a fixed singular point. The expressions for the second derivatives as $x \to 0$ represented an uncertainty of the form $o(x^2)/o(x^2)$. The numerical integration of such a system by steps in the neighborhood of the point $x = 0$ is impossible. The solution of this system in the neighborhood of a singular point $0 \leqslant x \leqslant x_1$. was obtained with the aid of a specialized program for actions with series. For $x > x_1$ the same system was resolved by the Adams method.

The expansion of the solution in converging power series is also possible in the neighborhood of mobile singular points of a certain form along the so-called exclusive directions determined by the character of the singularity.

In some boundary value problems (to which, for example, is reduced the solution of gas dynamics problem of supersonic flow past a blunt body by integration methods, see [3]) the condition at the second mobile end is the passage of the integral curve through a singular point (saddle type). This condition may be fulfilled by assorting a certain parameter given at the first end at solution of the Cauchy problem. The selection of the parameter is materialized as a result of sticking the numerical solution of the Cauchy problem at a certain distance from the singular point with the expansion of the solution in series in the neighborhood of the singular point. This"sticking together" is a rather delicate operation, for if we dispose of a rather small number of terms of the series, the sticking will have to be carried out in the direct neighborhood of the singular point, where significant errors may occur during the numerical integration by steps.

The utilization of a specialized program for actions with series allows us to dispose of a sufficient number of terms of series, so that such a "sticking together" may be conducted at a significant distance from the singular point, where the numerical integration by steps is sufficiently precise.

The specialized program may be also utilized during the compilation of tables for certain special functions constituting power series. Obviously, substitution of variables must be made beforehand in the equation, allowing us to separate the singularity from the solution. (For example, for the Bessel equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ such a substitution will be $y = x^p u$).

In conclusion the author expresses his gratitude to M. D. Ladyzhenskiy for useful discussions concerned with the application of a specialized program for concrete problems.

*** THE END ***

ADDENDUM

After the manuscript has been sent to printer, the author took cognizance of the work [4], where a program is worked out for the analytical differentiation of elementary functions, and on its basis the problem is resolved of series construction for finding the solution of a single differential equation resolved relative to the senior derivative.

Expounded in the present work are the principles of programming a method of indeterminate coefficients for systems of ordinary nonlinear differential equations of the type, when the coefficients of series, beginning from a certain number, are determined from systems of linear algebraic equations.

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